Strict Positivity of the Dirichlet Square Root Behind OEIS A317940

Dominic Dabish
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Abstract

Let g be the multiplicative arithmetic function determined by $g(p^e) = 2^{A005187(e)}$, where $A005187(e) = \sum_{k\geq 0}\lfloor e/2^k\rfloor$. If $f(1) = 1$ and $f * f = g$ under Dirichlet convolution, OEIS A317940 records the numerators of $f(n)$ and asks whether they are nonnegative. We prove the stronger statement $f(n) > 0$ for every $n \geq 1$. The proof reduces the problem to positivity of the formal square root of a binary Euler product. The exact Formal Conjectures statement has also been verified in Lean 4.

1 Local reduction

Because g is multiplicative, its normalized Dirichlet square root is multiplicative. Put $c_e = f(p^e)$. Then

$$\sum_{j=0}^{e} c_j c_{e-j} = 2^{A005187(e)}. \tag{1}$$

The recurrence $A005187(e) = e + A005187(\lfloor e/2\rfloor)$ yields $A005187(e) = 2e - s_2(e)$, where $s_2(e)$ is the binary digit sum. Set $b_e = 2^{-s_2(e)}$. Then $b_{2r} = b_r$ and $b_{2r+1} = b_r/2$.

2 The binary Euler product

Let $B(z) = \sum_{e\geq 0} b_e z^e$. Unique binary expansion gives

$$B(z) = \prod_{r\geq 0}\left(1 + \frac{z^{2^r}}{2}\right).$$

Let $A(z)^2 = B(z)$ with $A(0) = 1$, and write $A(z) = \sum a_e z^e$.

Lemma 1. *Every nonconstant coefficient of* $\log A(z)$ *is strictly positive.*

Proof. Write $n = 2^v m$ with m odd. Expanding the formal logarithm gives

$$[z^n]\log A(z) = \frac{1}{2m}\left(2^{-m} - \sum_{j=1}^{v} 2^{-j-2^j m}\right).$$

The negative sum is strictly smaller than $2^{-2m}\sum_{j\geq 1} 2^{-j} = 2^{-2m} < 2^{-m}$. □

Theorem 1. *Every coefficient* a_e *is strictly positive.*

Proof. Since $A = \exp(\log A)$ and every nonconstant coefficient of $\log A$ is positive, the coefficient of z^e in A has the positive linear contribution $[z^e]\log A$, while all higher exponential contributions are nonnegative. □

3 Return to Dirichlet convolution

Set $c_e = 4^e a_e$. Rescaling $A(z)^2 = B(z)$ by $z \mapsto 4z$ gives

$$\sum_{j=0}^{e} c_j c_{e-j} = 4^e b_e = 2^{A005187(e)},$$

so (1) holds and $c_e > 0$. Define the multiplicative arithmetic function h by $h(p^e) = c_e$. Then $h * h = g$. Separating the endpoint divisors in this equality gives, for $n > 1$,

$$h(n) = \frac{1}{2}\left(g(n) - \sum_{\substack{d \mid n \\ 1<d<n}} h(d)h(n/d) \right).$$

This is exactly the recursion defining the rational function behind A317940. By strong induction, $f = h$, and hence $f(n) > 0$ for all $n \geq 1$.

Theorem 2. *The rational function underlying OEIS A317940 is strictly positive at every positive integer. In particular, A317940 has no negative or zero signed numerator.*

4 Generalization

For $0 < q \leq 1$ and $\alpha > 0$, every coefficient of

$$\prod_{r \geq 0} (1 + qz^{2^r})^{\alpha}$$

is strictly positive. Indeed, for $n = 2^v m$ with m odd,

$$[z^n] \log \prod_{r \geq 0} (1 + qz^{2^r})^{\alpha} = \frac{\alpha}{m}\left(q^m - \sum_{j=1}^{v} \frac{q^{2^j m}}{2^j} \right) > 0.$$

Formal verification

A Lean 4 proof checks the exact theorem named `A317940_f_nonnegative` against the Google DeepMind Formal Conjectures definitions. The proof contains no placeholders or custom axioms. A complete upstream-repository build and axiom audit accompany the source.